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Superstability of Approximate Cosine Type Functions on the Monoid $\mathbb{R}^2$

B. Bouikhalene[1], E. Elqorachi[2], A. Charifi[1]

[1] *University Sultan Moulay Slimane, Laboratory LIRST, Polydisciplinary Faculty, Beni-Mellal, Morocco*

[2] *Ibn Zohr University, Faculty of Sciences, Agadir, Morocco*

bbouikhalene@yahoo.fr, elqorachi@yahoo.fr, charifi2000@yahoo.fr

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Abstract: In this paper, we study the superstability problem for the cosine type functional equation

$$f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2)$$

on the commutative monoid $(\mathbb{R}^2, \times)$. As a result we obtain cosine type functions satisfying the equation approximately.

Key words: Functional equation, cosine function, superstability, multiplicative function.

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1. Introduction

In 1940, S. M. Ulam [17] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Question 1.1. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric d. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x_1 \in G_1$?.

In 1941, Hyers [11] answered this question for the case where G_1 and G_2 are Banach spaces. In [2] and [15] Aoki and Th. M. Rassias respectively provided a generalization of Hyer's theorem which allows the Cauchy difference to be unbounded. The interested reader may refer to the book by Hyers, Isac, Rassias [12] for an in depth account on the subject of stability of functional equations. In 1982, J.M. Rassias [14] solved the Ulam problem by involving a product of powers of norms. Since then, the stability problems of various functional equations has been investigated by many authors (see [9], [10]). In [4] and [7] Baker et al. and Bourgin respectively, introduced the notion that by now is frequently referred to as superstability or Baker's stability : if a function f satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. The superstability of d'Alembert's functional equation $f(x+y)+f(x-y) = 2f(x)f(y)$ was investigated by Baker [5] and Cholewa [8]. Badora and Ger [3] proved its superstability under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$. In a previous work, Bouikhalene et al [6] investigated the superstability of the cosine functional equation on the Heisenberg group.

Now, Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the commutative monoid equipped with composition rule

$$(x_1, y_1)(x_2, y_2) := (x_1x_2, x_1y_2 + x_2y_1). \tag{1.1}$$

The map $i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, given by $i(x, y) = (x, -y)$ for any $(x, y) \in \mathbb{R}^2$, is an involution of $\mathbb{R}^2$, i.e., $i\big((x_1, y_1)(x_2, y_2)\big) = i(x_1, y_1)i(x_2, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $i{\circ}i = id$ (the identity map). Consider the functional equation

$$f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2) \tag{1.2}$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. By setting $a = (x_1, y_1)$, $b = (x_2, y_2)$ in (1.2) we obtain the cosine type functional equation

$$f(ab) + f\big(ai(b)\big) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2 \tag{1.3}$$

on the commutative monoid $\mathbb{R}^2$. This equation has the same form as the cosine functional equation, also called d'Alembert's functional equation ([1], [13])

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G, \tag{1.4}$$

on an abelian group G, except that the group inversion $y \longrightarrow -y$ is replaced by the involution i. We say that a function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is of approximate a cosine type function, if there is $\delta > 0$ such that

$$\big|f(ab) + f\big(ai(b)\big) - 2f(a)f(b)\big| < \delta, \quad a, b \in \mathbb{R}^2. \tag{1.5}$$

In the case where $\delta = 0$, f satisfies the functional equation (1.3). We call f a cosine type function on $\mathbb{R}^2$. The main purpose of this work is to prove the superstability problem of equation (1.2) in the commutative monoid $\mathbb{R}^2$.

2. Superstability of equation (1.2)

Proposition 2.1. *let* $\varphi, \psi, \phi, \zeta : \mathbb{R} \longrightarrow [0, +\infty[$ *be functions and let* $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ *satisfies the functional inequality*

$$\big|f(ab) + f\big(ai(b)\big) - 2f(a)f(b)\big| \le \min\big\{\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)\big\} \tag{2.1}$$

for any $a = (x_1, y_1)$, $b = (x_2, y_2) \in \mathbb{R}^2$. *Then* $m(x) = f(x, 0)$ *for any* $x \in \mathbb{R}$, *is either bounded or multiplicative function from* $\mathbb{R}$ *to* $\mathbb{C}$. *Furthermore* f *satisfies the following inequality*

$$\left|f(a)^2 - \frac{1}{2}f(a^2) - \frac{1}{2}m(x^2)\right| \le \frac{1}{2}\min\big\{\varphi(x), \psi(y), \phi(x), \zeta(y)\big\} \tag{2.2}$$

for any $a = (x, y) \in \mathbb{R}^2$.

Proof. Setting $a = (x, 0)$, $b = (y, 0)$ in (2.1), we get

$$\big|f(x,0)f(y,0) - f(xy,0)\big| \le \frac{1}{2}\min\big\{\varphi(x), \psi(0), \phi(y), \zeta(0)\big\}$$

for any $x, y \in \mathbb{R}$. According to [16] we get that $m(x) = f(x, 0)$ for any $x \in \mathbb{R}$ is either bounded or a multiplicative function from $\mathbb{R}$ to $\mathbb{C}$. Once again, putting $a = (x, y)$ in (2.1) we get that

$$\big|f(x^2, 2xy) + f(x^2, 0) - 2f(x, y)^2\big| \le \min\big\{\varphi(x), \psi(y), \phi(x), \zeta(y)\big\}$$

for any $x, y \in \mathbb{R}$. So that

$$\left|f(a)^2 - \frac{1}{2}f(a^2) - \frac{1}{2}m(x^2)\right| \le \frac{1}{2}\min\big\{\varphi(x), \psi(y), \phi(x), \zeta(y)\big\}$$

for any $a = (x, y) \in \mathbb{R}^2$. ∎

Proposition 2.2. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ *satisfies the functional inequality* (2.1) *and let* $F(y) = f(1, y)$ *for any* $y \in \mathbb{R}$. *Then*

i) *F is either bounded, or*

ii) *F satisfies the cosine functional equation*

$$F(x + y) + F(x - y) = 2F(x)F(y), \quad x, y \in \mathbb{R}. \tag{2.3}$$

Further, in the latter case, there exists an exponential function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ *such that*

$$F(x) = \frac{1}{2}\big(\gamma(x) + \gamma(-x)\big)$$

for any $x \in \mathbb{R}$. ▮

Proof. Let $a = (1, x)$, $b = (1, y)$ for any $x, y \in \mathbb{R}$ in (2.1). By setting $F(y) = f(1, y)$ for any $y \in \mathbb{R}$ we get

$$\big|F(x+y) + F(x-y) - 2F(x)F(y)\big| \leq \min\big\{\varphi(1), \psi(x), \phi(1), \zeta(y)\big\}$$

for any $x, y \in \mathbb{R}$. According to ([3], [5]) it follows that F is either bounded or F is a cosine function. In view of ([1], [5], [13]) we get that there exists an exponential function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ such that $F(x) = \frac{1}{2}\big(\gamma(x) + \gamma(-x)\big)$ for any $x \in \mathbb{R}$. ▮

Proposition 2.3. *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ *satisfies the functional inequality* (2.1)*. Then* f *is either bounded or* $f \circ i = f$*.*

Proof. Let $P_f = \frac{f + f \circ i}{2}$. Since f satisfies (2.1), we have

$$\big|P_f(ab) + P_f\big(ai(b)\big) - 2P_f(a)f(b)\big| \leq \min\big\{\varphi(x_1), \tilde{P}_\psi(y_1), \phi(x_2), \tilde{P}_\zeta(y_2)\big\}$$

for any $a = (x_1, y_1)$, $b = (x_2, y_2) \in \mathbb{R}^2$, where $\tilde{P}_\psi(x) = \frac{\psi(x) + \psi(-x)}{2}$ for any $x \in \mathbb{R}$. By using the same way as in [3] and [5] we get that f is either bounded or f satisfies the Wilson's type functional equation

$$P_f(ab) + P_f\big(ai(b)\big) = 2P_f(a)f(b), \quad a, b \in \mathbb{R}^2$$

on the commutative monoid $\mathbb{R}^2$. By small computations we get that $f \circ i = f$. ▮

Proposition 2.4. *Let* $\varphi, \psi, \phi, \zeta : \mathbb{R} \longrightarrow [0, +\infty[$ *be functions and let* $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$*, with* $f(0,0) \neq 0$*, satisfies the functional inequality* (2.1)*. Then* f *is bounded and we have*

$$\big|f(a) - 1\big| \leq \frac{1}{2|f(0,0)|} \min\big\{\varphi(x), \psi(y), \phi(0), \zeta(0)\big\} \tag{2.4}$$

for any $a = (x, y) \in \mathbb{R}^2$.

Proof. By letting $b = (0,0)$ in (2.1) we get

$$\big|2f(0,0) - 2f(a)f(0,0)\big| \le \min\big\{\varphi(x), \psi(y), \phi(0), \zeta(0)\big\}$$

for any $a = (x,y) \in \mathbb{R}$. So that we have

$$\big|2f(0,0)\big|\big|f(a) - 1\big| \le \min\big\{\varphi(x), \psi(y), \phi(0), \zeta(0)\big\}$$

for any $a = (x,y) \in \mathbb{R}^2$. ∎

THEOREM 2.5. *let $\varphi,\ \psi,\ \phi,\ \zeta : \mathbb{R} \longrightarrow [0,+\infty[$ be functions and let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ satisfies the functional inequality* (2.1). *Then*

i) *f is either bounded and*

$$\big|f(0,y)^2 - f(0,0)\big| \le \frac{1}{2}\min\big\{\varphi(0), \psi(y), \phi(0), \zeta(y)\big\} \tag{2.5}$$

for any $y \in \mathbb{R}$ or

ii) *f satisfies the functional inequality*

$$\left|f(a) - m(x)\frac{\gamma(\frac{y}{x}) + \gamma(\frac{-y}{x})}{2}\right| \le \frac{1}{2}\min\big\{\varphi(x), \psi(0), \phi(1), \zeta(\tfrac{y}{x})\big\} \tag{2.6}$$

for any $a = (x,y) \in \mathbb{R}$ with $x \neq 0$, where $m : \mathbb{R} \longrightarrow \mathbb{C}$ is a multiplicative function and $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ is an exponential function.

Proof. i) Letting $a = b = (0,y)$ in (2.1), we get

$$\big|f(0,y)^2 - f(0,0)\big| \le \frac{1}{2}\min\big\{\varphi(0), \psi(y), \phi(0), \zeta(y)\big\}$$

for any $y \in \mathbb{R}$.

ii) Let f be unbounded. Hence by Propositions 2.1 and 2.2 we get that $f(x,0) = m(x)$ for any $x \in \mathbb{R}$ is a multiplicative function from $\mathbb{R}$ to $\mathbb{C}$ and $f(1,y) = F(y)$ for any $y \in \mathbb{R}$ is a solution of the cosine functional equation (1.4). Therefore there exists an exponential function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ such that $f(1,y) = F(y) = \frac{\gamma(y)+\gamma(-y)}{2}$ for any $y \in \mathbb{R}$. By letting $a = (x,0), b = (1, \frac{y}{x})$, with $x \neq 0$, in (2.1) we get the following inequality

$$\big|f(x,y) + f(x,-y) - 2f(x,0)f(1,\tfrac{y}{x})\big| \le \min\big\{\varphi(x), \psi(0), \phi(1), \zeta(\tfrac{y}{x})\big\} \tag{2.7}$$

for any $x, y \in \mathbb{R}$ with $x \neq 0$. Therefore by Proposition 2.3 we get that $f(x,y) = f \circ i(x,y) = f(x,-y)$ for any $x, y \in \mathbb{R}$. So that we get from (2.7) that

$$\left|f(x,y) - m(x)F(\tfrac{y}{x})\right| \leq \frac{1}{2}\min\left\{\varphi(x), \psi(0), \phi(1), \zeta(\tfrac{y}{x})\right\}$$

for any $x, y \in \mathbb{R}$ with $x \neq 0$. ∎

In the next corollary we let $\varphi(x_1) = \psi(y_1) = \varphi(x_2) = \zeta(y_2) = \delta$ for any $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Corollary 2.6. *Let $\delta > 0$ and let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ satisfies the functional inequality*

$$\left|f(ab) + f(ai(b)) - 2f(a)f(b)\right| \leq \delta \tag{2.8}$$

for any $a, b \in \mathbb{R}^2$. Then

i) *f is bounded and there exists $\eta \in \mathbb{C}^*$ such that $|f(a) - 1| \leq \frac{\delta}{2\eta}$ for any $a = (x,y) \in \mathbb{R}$, with $x \neq 0$. Furthermore $|f(0,y) - \eta| \leq \frac{\delta}{2}$ for an $y \in \mathbb{R}$ or*

ii) *f is unbounded and there exist a multiplicative function $m : \mathbb{R} \longrightarrow \mathbb{C}$ and an exponential function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ such that*

$$\left|f(a) - m(x)\frac{\gamma(\frac{y}{x}) + \gamma(\frac{-y}{x})}{2}\right| \leq \frac{\delta}{2} \tag{2.9}$$

for any $a = (x,y) \in \mathbb{R}^2$ with $x \neq 0$.

Proof. By using Proposition 2.4 and Theorem 2.5 with $\eta = f(0,0)$. ∎

In the next corollary we give the explicit formula of cosine type functions on $\mathbb{R}^2$

Corollary 2.7. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a cosine type function on $\mathbb{R}^2$. Then*

i) *$f(x,y) = 1$ for any $x, y \in \mathbb{R}$ or*

ii)

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{m(x)}{2}\left(\gamma(\frac{y}{x}) + \gamma(-\frac{y}{x})\right) & \text{if } x \neq 0, \end{cases}$$

for any $x, y \in \mathbb{R}$.

Proof. By letting $\delta = 0$ in Corollary 2.6. ∎

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